INFINITE TYPE POWER SERIES SUBSPACES OF FINITE TYPE POWER SERIES SPACES

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ABSTRACT

Nuclear power series spaces of finite type, $\Lambda_0(\alpha)$, and infinite type, $\Lambda_\infty(\alpha)$, are considered. Sufficient conditions are given on α for which there exists a β such that $\Lambda_\infty(\beta)$ is isomorphic to a subspace of $\Lambda_0(\alpha)$ and also for which there does not exist such a β. In certain cases it is possible to take $\beta=\alpha$. The results in this papar are related to earlier results by S. Rolewicz and V. P. Zaharyuta.

One way of investigating the structure of nuclear Fréchet spaces would be to characterize, for a given space, all infinite dimensional closed subspaces, up to isomorphism. For example, in the case of the space ω of all infinite sequences of scalars, the answer is simply ω itself [9, (3.1)]. In this paper we consider a very limited version of this extremely general question. We take a given power series space of finite type (see below for definitions) and try to determine if it is possible for this space to have a subspace isomorphic to a power series space of infinite type. A more difficult next step will be to try to find out exactly which such spaces can appear as subspaces.

The first result in this direction was due to S. Rolewicz in 1961 who showed [12] that such an embedding can be possible. Some of our results are extensions of this observation, but inasmuch as Rolewicz used a representation in terms of spaces of analytic functions and we use only sequence space representations, our methods are quite different.

More recently, V. P. Zaharyuta has shown [13] that no power series space of infinite type can contain a subspace isomorphic to a power series space of finite type. Indeed, our main motivation in this paper has been to consider the statement of Zaharyuta's theorem with the words "infinite" and "finite" interchanged. In

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view of Rolewicz's result it is not surprising that this transposition puts us into an entirely new situation. We show, for example, that Zaharyuta's method, consideration of compact maps, is entirely inapplicable (Theorem 5) and the fact that the answer to our question is sometimes yes and sometimes no suggests that the structure of finite type power series spaces is more complicated than that of infinite type spaces.

Usually, the most useful tool in studying nuclear Fréchet spaces is the fact that the seminorms defining the topology can be taken to be seminorms which come from inner products and hence the theory of Hilbert spaces can be applied (see [10], for example). In this paper, we exploit the fact that the role played by l_2 can also be played by l_∞. Considering only elements which can be written as finite linear combinations of elements of a basis gives a combinatorial flavor to the problem of estimating the seminorm of an element of our space.

The major technical result in this paper is Theorem 1 which gives several characterizations of when a certain subspace of a given finite type power series space is isomorphic to a space of infinite type. Theorems 3 and 4 give new classes of finite type power series spaces which have infinite type subspaces and Theorem 2 provides a limitation on the kind of embedding which is possible. Among the consequences of these results are two interesting new facts. First we show that quasi-equivalent bases (see [10] for definitions) are not the same if one considers all possible subspaces which can be generated by block basic sequences. Secondly, we give the first example of a block basic sequence which generates a noncomplemented subspace and has no extension to a basis (cf. [5] and [8]). In Theorem 6 we show that there are some power series spaces of finite type which have no subspace isomorphic to a power series space of infinite type.

The term scalars will refer to real or complex numbers; subspace will mean closed subspace and isomorphic will mean linearly isomorphic.

The symbol $\mathbb{N}$ will stand for the positive integers and we will variously denote a sequence by α, (α_n), $(\alpha_n)_n$. Sometimes we will refer to an infinite subset $\mathbb{N}_0 \subset \mathbb{N}$ and then use $\mathbb{N}_0$ to index a sequence. We mean, in this case, that $\mathbb{N}_0$ is to be considered in its natural order as a subsequence of $\mathbb{N}$.

The n'th *coordinate sequence* e_n is the sequence which is 1 in its n'th coordinate and 0 elsewhere.

We recall that a sequence (x_n) in a locally convex space E is a *basis* if for each $x \in E$ there is a unique sequence (t_n) of scalars such that

$$x = \sum_n t_n x_n.$$

A sequence in E is a *basic sequence* if it is a basis for the subspace which it generates. If E is a locally convex space and (x_n) is a basis then a *block basic sequence of* (x_n) is a sequence (y_n) determined by a strictly increasing sequence, $0 = p_0 < p_1 < \cdots$ of integers and a sequence (t_n) of scalars via the relation

$$y_n = \sum_{i=p_{n-1}+1}^{p_n} t_i x_i \neq 0, \qquad n = 1, 2, \cdots.$$

If E is a Fréchet space then a block basic sequence is always a basic sequence.

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Preliminary results

We begin by recalling some of the basic facts about power series spaces. For more details see [11]. We define an *exponent sequence* to be a nondecreasing sequence $\alpha = (\alpha_n)$ of nonnegative numbers. Given α we define the corresponding *power series spaces* $\Lambda_0(\alpha)$, $\Lambda_\infty(\alpha)$ *of finite and infinite type* respectively by

$$\Lambda_0(\alpha) = \left\{\xi = (\xi_n) \colon p_k(\xi) = \sum_n |\xi_n| \left(\frac{k}{k+1}\right)^{\alpha_n} < \infty,\ k = 1, 2, \cdots\right\}$$

$$\Lambda_\infty(\alpha) = \left\{\xi = (\xi_n) \colon p_k(\xi) = \sum_n |\xi_n| k^{\alpha_n} < \infty,\ k = 1, 2, \cdots\right\}.$$

In each case the functions p_k, $k = 1, 2, \cdots$ form a fundamental system of seminorms for Fréchet space topologies on $\Lambda_0(\alpha)$, $\Lambda_\infty(\alpha)$.

An exponent sequence α is said to be a *nuclear exponent sequence of finite type* if

$$\sum_n k^{-\alpha_n} < \infty \text{ for all } k = 1, 2, \cdots$$

and a *nuclear exponent sequence of infinite type* if

$$\sum_n k^{-\alpha_n} < \infty \text{ for some } k = 1, 2, \cdots.$$

The basic properties of power series spaces which we will need are contained in the following proposition. They are well known and easy to derive from the definitions.

PROPOSITION 1. *Let α be an exponent sequence.*

(i) *α is a nuclear exponent sequence of finite, respectively infinite, type iff the Fréchet space $\Lambda_0(\alpha)$, respectively $\Lambda_\infty(\alpha)$, is a nuclear space. In this case the fundamental system of seminorms (p_k) can be replaced by $(\tilde{p}_k)$ where*

$$\tilde{p}_k(\xi) = \sup_n |\xi_n| \left(\frac{k}{k+1}\right)^{\alpha_n},$$

respectively

$$\tilde{p}_k(\xi) = \sup_n |\xi_n| k^{\alpha_n}.$$

Moreover, in this case, the coordinate sequence (e_n) forms a basis for the space, called the coordinate basis.

(ii) *If α is a nuclear exponent sequence, then $\lim_n \alpha_n = \infty$.*

(iii) *If $\sum_n 1/\alpha_n < \infty$ then α is a nuclear exponent sequence of both finite and infinite type.*

(iv) *Every nuclear exponent sequence of finite type is also a nuclear exponent sequence of infinite type but not conversely. A counter-example is given by $\alpha_n = \log n$.*

(v) *If α and β are two nuclear exponent sequences of the same type and γ is the sequence obtained by rearranging the sequence $(\alpha_1, \beta_1, \alpha_2, \beta_2, \cdots)$ into a nondecreasing sequence, then γ is a nuclear exponent sequence of the same type.*

(vi) *If α is a nuclear exponent sequence and $c > 0$ then the sequence $(c\alpha_n)_n$ is a nuclear exponent sequence of the same type.*

(vii) *Any subsequence of a nuclear exponent sequence is a nuclear exponent sequence of the same type.*

Now let (a_n^k) be a fixed infinite matrix of scalars satisfying the following conditions for all n, k:

$$0 < a_n^k \leqq a_n^{k+1} \tag{1}$$

$$\frac{a_n^k}{a_{n+1}^k} \geqq \frac{a_n^{k+1}}{a_{n+1}^{k+1}}. \tag{2}$$

For a sequence of scalars $t_{p_0+1}, \cdots, t_{p_1}$ $(p_0 < p_1)$ and for each $k = 1, 2, \cdots$ we define

$$q^k(t_{p_0+1}, \cdots, t_{p_1}) = \max\left\{q : \max_{p_0 < i \leqq p_1} |t_i| a^k = |t_q| a_q^k\right\}.$$

LEMMA 1. *In the context of the above notation we have*

$$q^k(t_{p_0+1},\cdots,t_{p_1}) \leqq q^{k+1}(t_{p_0+1},\cdots,t_{p_1}).$$

PROOF. If we have $p_0 < i < j \leqq p_1$ with

$$|t_i|a_i^k \leqq |t_j|a_j,$$

then

$$\frac{a_i^{k+1}}{a_j^{k+1}} \leqq \frac{a_i^k}{a_j^k} \leqq \frac{|t_j|}{|t_i|}$$

so that

$$|t_i|a_i^{k+1} \leqq |t_j|a_j^{k+1}.$$

This implies that if $q^k(t_{p_0+1},\cdots,t_{p_1}) = j$ then $q^{k+1}(t_{p_0+1},\cdots,t_{p_1}) \geqq j$ and the lemma is proved.

Suppose we think of $t_{p_0+1},\cdots,t_{p_1}$ as fixed for the moment and k as running through consecutive integers $1,2,\cdots,K$. Then the numbers $q^k = q^k(t_{p_0+1},\cdots,t_{p_1})$ run through the integers $p_0+1,\cdots,p_1$ with perhaps some omissions and/or some repetitions. Lemma 1 states that the sequence $q^1,\cdots,q^K$ must be nondecreasing. In Lemma 2 we show that, with an additional condition on the original matrix (a_n^k), these are the only restrictions and that, otherwise, any preassigned sequence $q^1,\cdots,q^K$ can be attained by an appropriate choice of $t_{p_0+1},\cdots,t_{p_1}$. This result will be crucial in all of our constructions of subspaces.

LEMMA 2. *Assume that the inequality* (2) *is strict for all* n,k. *Let* $\rho_0,\cdots,\rho_m$ *be any sequence of integers such that*

$$0 < \rho_0 < \rho_1 < \cdots < \rho_m, \quad m \leqq p_1 - p_0,$$

and let $q^1,\cdots,q^m$ *be any integers which satisfy*

$$p_0 < q^1 < q^2 < \cdots < q^m \leqq p_1.$$

Then there exist scalars $t_{p_0+1},\cdots,t_{p_1}$ *such that*

$$q^k(t_{p_0+1},\cdots,t_{p_1}) = q^l \text{ for } \rho_{l-1} < k \leqq \rho_l, \ l = 1,2,\cdots,m.$$

PROOF. Set $t_i = 0$ for $i \neq q^1,\cdots,q^m$ and set $t_{q^1} = 1$. Then we choose t_{q^j}, $j = 2,\cdots,m$ inductively to satisfy the inequality

$$\text{(3)} \qquad \max_{\rho_{j-1}<k\leqq\rho_j} |t_{q^{j-1}}| \frac{a_{q^{j-1}}^k}{a_{q^j}^k} < |t_{q^j}| < \min_{\rho_{j-2}<k\leqq\rho_{j-1}} |t_{q^{j-1}}| \frac{a_{q^{j-1}}^k}{a_{q^j}^k}.$$

This choice is possible because our hypotheses that the inequality in (2) is strict and that $q^{j-1} < q^j$ imply that every term on the far left of inequality (3) is strictly less than every term on the far right.

Now fix $l = 1, \cdots, m-1$ and first consider some j with $l < j \leqq m$. By (3) we have

$$|t_{q^j}|\, a^k_{q^j} < |t_{q^{j-1}}|\, a^k_{q^{j-1}} \text{ for } \rho_{j-2} < k \leqq \rho_{j-1}.$$

We claim that this inequality also holds for $\rho_{l-1} < k \leqq \rho_l$. Indeed, using it with $k = \rho_{j-1}$ and the fact that $\rho_l \leqq \rho_{j-1}$ we obtain for $\rho_{l-1} < k \leqq \rho_l$,

$$\frac{|t_{q^j}|}{|t_{q^{j-1}}|} \leqq \frac{a^{\rho_{j-1}}_{q^{j-1}}}{a^{\rho_{j-1}}_{q^j}} \leqq \frac{a^{\rho_l}_{q^{j-1}}}{a^{\rho_l}_{q^j}} \leqq \frac{a^k_{q^{j-1}}}{a^k_{q^j}}.$$

Thus we have proved

$$|t_{q^j}|\, a^k_{q^j} < |t_{q^{j-1}}|\, a^k_{q^{j-1}} \text{ for } l < j \leqq m \text{ and } \rho_{l-1} < k \leqq \rho_l.$$

Repeating this inequality with j replaced successively by $j-1, \cdots, l+1$ we obtain

$$|t_{q^j}|\, a^k_{q^j} < |t_{q^l}|\, a^k_{q^l} \text{ for } l < j \leqq m \text{ and } \rho_{l-1} < k \leqq \rho_l.$$

From the definitions we immediately conclude that

$$q^k(t_{p_0+1}, \cdots, t_{p_1}) \leqq q^l \text{ for } \rho_{l-1} < k \leqq \rho_l.$$

For $l = m$ this inequality follows immediately from the fact that $|t_i| \geqq 0$ for all i and $|t_i| \neq 0$ iff $i = q^1, \cdots, q^m$.

To obtain the reverse inequality we argue analogously. Suppose that $l > 1$ and we have $1 \leqq j < l$. Applying (3) we obtain

$$|t_{q^j}|\, a^k_{q^j} < |t_{q^{j+1}}|\, a^k_{q^{j+1}} \text{ for } \rho_j < k \leqq \rho_{j+1}.$$

Again we wish to extend this inequality to the case $\rho_{l-1} < k \leqq \rho_l$. Using it with $k = \rho_j + 1$ and the fact that $\rho_j \leqq \rho_{l-1}$ we obtain for $\rho_{l-1} < k \leqq \rho_l$,

$$\frac{|t_{q^{j+1}}|}{|t_{q^j}|} > \frac{a^{\rho_j+1}_{q^j}}{a^{\rho_j+1}_{q^{j+1}}} \geqq \frac{a^{\rho_{l-1}+1}_{q^j}}{a^{\rho_{l-1}+1}_{q^{j+1}}} \geqq \frac{a^k_{q^j}}{a^k_{q^{j+1}}}.$$

Thus we have proved

$$|t_{q^j}|\, a^k_{q^j} < |t_{q^{j+1}}|\, a^k_{q^{j+1}} \text{ for } 1 \leqq j < l \text{ and } \rho_{l-1} < k \leqq \rho_l.$$

Repeating this inequality with j replaced successively by $j+1, \cdots, l-1$, we obtain

$$|t_{q^j}|\, a^k_{q^j} < |t_{q^l}|\, a^k_{q^l} \text{ for } 1 \leqq j < l \text{ and } \rho_{l-1} < k \leqq \rho_l.$$

This shows that

$$q_k(t_{p_0+1},\cdots,t_{p_1} \geqq q^l \text{ for } \rho_{l-1} < k \leqq \rho_l.$$

Finally, for $l = 1$, this inequality follows as the first one did for $l = m$. Thus we have our two inequalities and the proof is completed.

For the remainder of the paper we specialize the matrix in the above discussion to be

$$a_n^k = \left(\frac{k}{k+1}\right)^{\alpha_n}, \ n,k = 1,2,\cdots,$$

where α is a nuclear exponent sequence of finite type. It is easy to check that the inequalities (1) and (2) hold, so Lemma 1 is valid. Moreover, the inequality (2) is strict so that Lemma 2 holds iff α is strictly increasing.

Now we want to introduce some auxiliary quantities based on the following parameters which may vary throughout the paper but will be fixed in any given discussion:

α — a nuclear exponent sequence of finite type

β — a nuclear exponent sequence of infinite type

(y_n) — a basic sequence in $\Lambda_0(\alpha)$ which is of the form

$$y_n = \sum_{i=1}^{p_n} t_i^n e_i, \ p_n < \infty, \ n = 1,2,\cdots$$

(d_n) — a sequence of nonzero scalars

π — a permutation of $\mathbb{N}$.

Then we define, for $n, k = 1,2,\cdots$,

$$q_n^k = q^k(t_1^n,\cdots,t_{p_n}^n)$$

$$\gamma_n^k = \frac{\alpha_{q_n^k}}{\beta_{\pi(n)}}$$

$$\mu_n^k = d_n^{1/\beta_{\pi(n)}} \, |t_{q_n^k}^n|^{1/\beta_{\pi(n)}} \left(\frac{k}{k+1}\right)^{\gamma_n^k}$$

$$r_n^k = \frac{\mu_n^{k+1}}{\mu_n^k}$$

Y = subspace of $\Lambda_0(\alpha)$ generated by (y_n).

First we derive a useful estimate of the quantity r_n^k.

LEMMA 3. *In the context of the proceeding notation, we have, for all n, k,*

$$\left(\frac{k^2+2k+1}{k^2+2k}\right)^{\gamma_n^k} \leqq r_n^k \leqq \left(\frac{k^2+2k+1}{k^2+2k}\right)^{\gamma_n^{k+1}}.$$

PROOF. From the definition of q_n^k we have

$$|t_{q^{k+1}}^n| \left(\frac{k}{k+1}\right)^{\alpha_{q_n^{k+1}}} \leqq |t_{q_n^k}^n| \left(\frac{k}{k+1}\right)^{\alpha_{q_n^k}}$$

and

$$|t_{q_n^k}^n| \left(\frac{k+1}{k+2}\right)^{\alpha_{q_n^k}} \leqq |t_{q_n^{k+1}}^n| \left(\frac{k+1}{k+2}\right)^{\alpha_{q_n^{k+1}}}$$

so

$$\left(\frac{k+1}{k+2}\right)^{\gamma_n^k-\gamma_n^{k+1}} \leqq \frac{d_n^{1/\beta_{\pi(n)}}|t_{q_n^{k+1}}^n|^{1/\beta_{\cdot}(n)}}{d_n^{1/\beta_{\pi(n)}}|t_{q_n^k}^n|^{1/\beta_{\pi(n)}}} \leqq \left(\frac{k}{k+1}\right)^{\gamma_n^k-\gamma_n^{k+1}}$$

and

$$\left(\frac{k+1}{k+2}\right)^{\gamma_n^k}\left(\frac{k+1}{k}\right)^{\gamma_n^k} \leqq \frac{\mu_n^{k+1}}{\mu_n^k} \leqq \left(\frac{k+1}{k+2}\right)^{\gamma_n^{k+1}}\left(\frac{k+1}{k}\right)^{\gamma_n^{k+1}}$$

which is exactly the desired inequality and the lemma is proved.

For the remainder of the paper we will be concerned with the problem of deciding when, for a given α, there exists (y_n) and a β such that Y is isomorphic to $\Lambda_\infty(\beta)$. As we will see, this amounts to being able to choose (d_n) and π so that the quantities defined above satisfy certain analytic conditions. The major technical tools which we will use to obtain such results are contained in the following characterization.

THEOREM 1. *In the context of the preceding notation, the following are equivalent.*

(i) *Y is isomorphic to $\Lambda_\infty(\beta)$.*

(ii) *There exist (d_n) and π such that*

(a) $\forall j \ \exists k$ *and* $M > 0 \ni$

$$j^{\beta_{\pi(n)}} \leqq M d_n |t_{q_n^k}^n| \left(\frac{k}{k+1}\right)^{\alpha_{q_n^k}} \text{ for all } n$$

and

(b) $\forall k \ \exists l$ *and* $N > 0 \ni$

$$d_n | t^n_{q^k_n} | \left(\frac{k}{k+1}\right)^{\alpha_{q^k_n}} \leqq N l^{\beta_{\pi(n)}} \textit{ for all } n.$$

(iii) *There exist* (d_n) *and* π *such that*

(a) $\overline{\lim_n} \mu^k_n < \infty$ *for each* k

and

(b) $\sup_k \underline{\lim_n} \mu^k_n = \infty.$

(iv) *There exists* π *such that*

(a) $\sup_n \gamma^k_n < \infty$ *for all* k

and

(b) $\sup_k \lim_n \sum_{j=1}^{k} \frac{\gamma^j_n}{j^2} = \infty.$

PROOF.

(i) $\Leftrightarrow$ (ii). According to the famous theorem of Dragilev [3], the isomorphism of Y and $\Lambda_\infty(\beta)$ is equivalent to the existence of (d_n) and π such that the bases (y_n) and $(d_n e_{\pi(n)})$ are equivalent, where (e_n) is the coordinate basis for $\Lambda_\infty(\beta)$. This is the same as saying that the sequence spaces determined by these bases are the same. Thus by a standard argument using the absolute basis theorem of Dynin and Mitiagin [6] and Proposition 1(i), the isomorphism is equivalent to the relation

$$\bigcap_k \frac{1}{c^k} \cdot l_1 = \bigcap_k \frac{1}{b^k} \cdot l_1$$

where, for each k, $b^k = (b^k_n)$ and $c^k = (c^k_n)$ are the sequences given by

$$b^k_n = | t^n_{q^k_n} | \left(\frac{k}{k+1}\right)^{\alpha_{q^k_n}},$$

$$c^k_n = \frac{1}{d_n} k^{\beta_{\pi(n)}}.$$

Here the notation, $(1/x) \cdot l_1$, refers to the set obtained by multiplying each sequence in l_1 coordinatewise by the sequence $(1/x_n)$.

Now the two sequence spaces mentioned above are echelon spaces in the sense of Köthe [7, §30] so they are equal iff their corresponding co-echelon spaces are also equal. Therefore, the relation

$$\bigcup_k c^k \cdot l_\infty = \bigcup_k b^k \cdot l_\infty$$

is equivalent to (i). But this equality is clearly equivalent to:

$$\text{(a) } \forall j \,\exists\, k \text{ and } M > 0 \ni c_n^j \leqq M b_n^k \quad \forall n$$

and

$$\text{(b) } \forall k \,\exists\, l \text{ and } N > 0 \ni b_n^k \leqq N c_n^l \quad \forall n.$$

Substituting for b^k and c^k transforms the above two statements exactly into (ii).

(ii)$\Leftrightarrow$(iii). If we raise both sides of the inequalities (ii) to the power $1/\beta_{\pi(n)}$, then they become

$$\text{(a) } \forall j \,\exists\, k \text{ and } M > 0 \ni j \leqq M^{1/\beta_{\pi(n)}} \mu_n^k \text{ for all } n$$

and

$$\text{(b) } \forall k \,\exists\, l \text{ and } N > 0 \ni \mu_n^k \leqq N^{1/\beta_{\pi(n)}} l \text{ for all } n.$$

We then take the limit inferior in (a), the limit superior in (b), and apply Proposition 1(ii) to obtain

$$\text{(a) } \forall j \,\exists\, k \ni j \leqq \varliminf_n \mu_n^k,$$

and

$$\text{(b) } \forall k \,\exists\, l \ni \varlimsup_n \mu_n^k \leqq l.$$

Clearly (a) is equivalent to (iii) (b) and (b) is equivalent to (iii) (a).

(iii) $\Rightarrow$ (iv). Choose some (d_n) and π such that (iii) (a) and (b) hold. First we prove (iv) (a). If this were not true then we would have some k_0 and an infinite subset $\mathbb{N}_0 \subset \mathbb{N}$ such that $\lim_{n\in N_0}\gamma_n^{k_0} = \infty$. Applying Lemma 1 for each n it would follow that $\lim_{n\in N_0}\gamma_n^k = \infty$ for all $k \geqq k_0$. By Lemma 3 this implies that $\lim_{n\in N_0} r_n^k = \infty$ for all $k \geqq k_0$. By (iii) (a), $\varlimsup_n \mu_n^{k+1} < \infty$ for $k \geqq k_0$ and so $\lim_{n\in N_0}\mu_n^k = 0$ for $k \geqq k_0^k$ and hence $\varliminf_n \mu_n^k = 0$ for $k \geqq k_0$. Obviously, in view of (iii) (a) applied with $k = 1, \cdots, k_0 - 1$ we have a contradiction of (iii) (b).

Next we prove (iv) (b). Applying Lemma 3 we have, for each n, k

$$\mu_n^k = \mu_n^1 r_n^1 \cdots r_n^{k-1} \leqq \mu_n^1 \prod_{j=1}^{k-1} \left(\frac{j^2 + 2j + 1}{j^2 + 2j}\right)^{\gamma_n^{j+1}}$$

so that for all k,

$$\varliminf_n \mu_n^k \leqq \left(\varlimsup_n \mu_n^1\right) \varliminf_n \left(\prod_{j=1}^{k-1} \left(\frac{j^2 + 2j + 1}{j^2 + 2j}\right)^{\gamma_n^{j+1}}\right).$$

Hence if, in view of (iii) (a) we set $M = \overline{\lim}_n \mu_n^1 < \infty$, it follows from the monotonicity of the logarithm function that

$$\log(\underline{\lim}_n \mu_n^k) \leqq \log M + \underline{\lim}_n \left(\sum_{j=1}^{k-1} \frac{\gamma_n^{j+1}}{(j+1)^2} (j+1)^2 \log\left(\frac{j^2+2j+1}{j^2+2j}\right)\right).$$

Now it is easy to check, using elementary calculus, that

$$\lim_{j\to\infty} (j+1)^2 \log\left(\frac{j^2+2j+1}{j^2+2j}\right) = 1$$

and so we have by (iii) (b),

$$\begin{aligned}
\infty &= \sup_k \log(\underline{\lim}_n \mu_n^k) \\
&\leqq \log M + \sup_k \underline{\lim}_n \left(\sum_{j=1}^{k-1} \frac{\gamma_n^{j+1}}{(j+1)^2} (j+1)^2 \log\left(\frac{j^2+2j+1}{j^2+2j}\right)\right) \\
&\leqq \log M + \sup_{1\leqq j<\infty} \left((j+1)^2 \log\left(\frac{j^2+2j+1}{j^2+2j}\right)\right) \sup_k \underline{\lim}_n \sum_{j=1}^{k-1} \frac{\gamma_n^{j+1}}{(j+1)^2},
\end{aligned}$$

and since both $\log M$ and the quantity

$$\sup_{1\leqq j<\infty} (j+1)^2 \log \frac{j^2+2j+1}{j^2+2j}$$

are finite, it follows that

$$\sup_k \underline{\lim}_n \sum_{j=1}^{k-1} \frac{\gamma_n^{j+1}}{(j+1)^2} = \infty,$$

which, along with (iv) (a) applied with $k = 1$, establishes (iv) (b).

(iv) $\Rightarrow$ (iii). Choose π such that (iv) (a) and (b) hold. Since $y_n \neq 0$ it follows from the definition of q_n^k that $t^n_{q_n^k} \neq 0$ for each k, n. Hence we can choose $d_n > 0$ such that $\mu_n^1 = 1$ for all n. By (iv) (a) we can set $M^k = \sup_{j\leqq k} \sup_n \gamma_n^j < \infty$ and apply Lemma 3 to obtain, for each n, k,

$$\begin{aligned}
\mu_n^k &= \mu_n^1 r_n^1 \cdots r_n^{k-1} \leqq \prod_{j=1}^{k-1} \left(\frac{j^2+2j+1}{j^2+2j}\right)^{\gamma_n^{j+1}} \\
&\leqq \left(\prod_{j=1}^{k-1} \frac{j^2+2j+1}{j^2+2j}\right)^{M^k} = \left(\frac{2k-2}{k}\right)^{M^k}
\end{aligned}$$

which establishes (iii) (a).

To obtain (iii) (b) we apply Lemma 3 again to obtain, for each n, k,

$$\mu_n^{k+1} = \mu_n^1 r_n^1 \cdots r_n^k \geqq \prod_{j=1}^{k} \left(\frac{j^2+2j+1}{j^2+2j}\right)^{\gamma_n^j}.$$

Hence there exists $\delta > 0$ such that for each n, k,

$$\begin{aligned}\log \mu_n^{k+1} &\geqq \prod_{j=1}^{k} \frac{\gamma_n^j}{j^2} j^2 \log\left(\frac{j^2+2j+1}{j^2+2j}\right) \\ &\geqq \delta \prod_{j=1}^{k} \frac{\gamma_n^j}{j^2}\end{aligned}$$

so

$$\sup_k \ \underline{\lim}_n \ \log \mu_n^{k+1} \geqq \delta \sup_k \ \underline{\lim}_n \ \sum_{j=1}^{k} \frac{\gamma_n^j}{j^2} = \infty$$

which establishes (iii) (b). This completes the proof of the theorem.

Condition (iv) in the preceding theorem is interesting because it shows explicitly the relative unimportance of the sequence (d_n). This condition may also be useful in deciding that specific values of β do, or do not, lead to infinite type power series spaces that can be embedded in $\Lambda_0(\alpha)$.

The following consequence of Theorem 1 lists some necessary conditions for the isomorphism of Y and $\Lambda_\infty(\beta)$. They will be useful later on.

PROPOSITION 2. *If Y is isomorphic to $\Lambda_\infty(\beta)$ and π is chosen so that condition* (iv) *of Theorem* 1 *holds, then there exists k_0 such that we have, for $k \geqq k_0$,*

$$\underline{\lim}_n \ \gamma_n^k > 0$$

and

$$\lim_n \ q_n^k = \infty.$$

PROOF. Suppose the first statement were false so that we had an infinite set K such that $\underline{\lim}_n \gamma_n^k = 0$ for all $k \in K$. Then by Lemma 1 and the definition of γ_n^k it would follow that $\underline{\lim}_n \gamma_n^k = 0$ for all k.

Now fix k so that we have an infinite subset $\mathbb{N}_0 \subset \mathbb{N}$ such that $\lim_{n \in N_0} \gamma_n^k = 0$. By Lemma 1 again it follows that $\lim_{n \in N_0} \gamma_n^j = 0$ for $j \leqq k$. Hence we have

$$\underline{\lim}_n \ \sum_{j=1}^{k} \frac{\gamma_n^j}{j^2} = 0.$$

Since this holds for each k we have a contradiction of Theorem 1 (iv).

The second statement follows from the first. In view of Lemma 1, it suffices to prove this statement for $k = k_0$. If it were false then we would have an infinite subset $\mathbb{N}_0 \subset \mathbb{N}$ such that $\sup_{n \in N_0} q_n^{k_0} < \infty$. But then, since $\lim_n \beta_{\pi(n)} = \infty$ (Proposition 1(ii)), it follows that

$$\lim_{n \in N_0} \gamma_n^{k_0} = \lim_{n \in N_0} \frac{\alpha_{q_n^{k_0}}}{\beta_{\pi(n)}} = 0$$

and this contradicts the first statement. Hence the second statement is also proved.

Main results

The first result on embedding infinite type power series spaces as subspaces of finite type power series spaces was due to S. Rolewicz [12] who showed how to embed $\Lambda_\infty(\alpha)$ as a subspace of $\Lambda_0(\alpha)$ when $\alpha_n = n$. Actually, he proved that the space of entire functions (in one complex variable) is isomorphic to a subspace of the space of functions analytic in the interior of a disk. This is the same thing since, as is well known, these two spaces are isomorphic to $\Lambda_\infty(\alpha)$, $\Lambda_0(\alpha)$ respectively when $\alpha_n = n$.

Rolewicz's result makes use of complex function theory and as such the actual construction is not obvious. Although B.S. Mitiagin has pointed out a much simpler argument, it is still not clear what the subspace looks like from a sequence space point of view. Our first main result shows that there is a limitation on how simple the embedding can be. It follows from our theorem and Rolewicz's result that not every subspace of a finite type power series space can be isomorphic to a subspace generated by a block basic sequence of the coordinate basis (e_n).

THEOREM 2. *If α is a nuclear exponent sequence of finite type which satisfies the condition*

$$\sup_n \frac{\alpha_{n+1}}{\alpha_n} < \infty,$$

then $\Lambda_\infty(\alpha)$ is not isomorphic to the subspace of $\Lambda_0(\alpha)$ generated by any block basic sequence of (e_n).

PROOF. By Proposition 1(iv), α is also of infinite type so the discussion of the preceding section is applicable. If we assume that (y_n) is a block basic sequence of (e_n) then we may assume that there exists a sequence $0 = p_0 < p_1 < \cdots$ and

$$y_n = \sum_{i=p_{n-1}+1}^{p_n} t_i e_i, \qquad n = 1, 2, \cdots.$$

This implies that for all n, k, j we have

$$p_{n-1} < q_n^k \leqq p_n < q_{n+1}^j$$

so

$$\alpha_{p_{n-1}} \leqq \alpha_{q_n^k} \leqq \alpha_{p_n} \leqq \alpha_{q^j_{n+1}}$$

Next we observe that since π is a permutation, there exists an infinite subset $\mathbb{N}_0 \subset \mathbb{N}$ such that $\pi(n+1) \leqq \pi(n) + 1$ for all $n \in \mathbb{N}_0$. Indeed if not, then there would be an integer n_0 such that $\pi(n+1) > \pi(n) + 1$ for all $n \geqq n_0$. Then we would have, for each $n_1 \geqq n_0$, that $\pi(n) \neq \pi(n_1) + 1$ for $n \geqq n_0$. That is, the infinitely many integers $\pi(n_0) + 1$, $\pi(n_0 + 1) + 1, \cdots$ would not be in the set $\{\pi(n) \colon n \geqq n_0\}$ and this would contradict the fact that π is onto.

Hence we may compute,

$$\begin{aligned}
\sup_k \sup_{n \in N} \gamma_n^k &= \sup_k \sup_{n \in N_0} \frac{\alpha_{q_n^k}}{\alpha_{\pi(n)}} \leqq \sup_k \sup_{n \in N_0} \frac{\alpha_{p_n}}{\alpha_{\pi(n)}} \\
&\leqq \sup_k \sup_{n \in N_0} \frac{\alpha_{q_{n+1}^1}}{\alpha_{\pi(n+1)}} \frac{\alpha_{\pi(n+1)}}{\alpha_{\pi(n)}} \\
&\leqq \left(\sup_{n \in N} \gamma_{n+1}^1\right) \sup_{n \in N_0} \frac{\alpha_{\pi(n)+1}}{\alpha_{\pi(n)}} \\
&\leqq \left(\sup_{n \in N} \gamma_n^1\right) \sup_{n \in N} \frac{\alpha_{n+1}}{\alpha_n}.
\end{aligned}$$

Now if we assume that condition (iv) of Theorem 1 holds then it follows from (iv) (a) and our hypotheses that this last expression is finite and so we have

$$\sup_k \varliminf_n \sum_{j=1}^k \frac{\gamma_n^j}{j^2} \leqq \left(\sup_{n \in N} \gamma_n^1\right) \left(\sup_{n \in N} \frac{\alpha_{n+1}}{\alpha_n}\right) \sum_{j=1}^\infty \frac{1}{j^2} < \infty$$

which contradicts (iv) (b) and the theorem is proved.

REMARK 1. In Theorem 2 it is not necessary to assume that $\alpha = \beta$ or that (α_{n+1}/α_n) is bounded. It is enough to assume that there exists an infinite subset $\mathbb{N}_0 \subset \mathbb{N}$ such that

$$\sup_{n \in N_0} \frac{\beta_{\pi(n+1)}}{\beta_{\pi(n)}} < \infty.$$

Then we could compute

$$\begin{aligned}
\sup_k \sup_{n \in N_0} \gamma_n^k &= \sup_k \sup_{n \in N_0} \frac{\alpha_{q_n^k}}{\beta_{\pi(n)}} \\
&\leqq \left(\sup_n \gamma_n^1\right) \sup_{n \in N_0} \frac{\beta_{\pi(n+1)}}{\beta_{\pi(n)}},
\end{aligned}$$

and the remainder of the argument is the same. This gives a necessary condition on β and π jointly for an embedding of $\Lambda_\infty(\beta)$ as a subspace of $\Lambda_0(\alpha)$ generated by a block basic sequence of (e_n).

In the next result we show that it *can* happen that $\Lambda_\infty(\alpha)$ is isomorphic to a subspace of $\Lambda_0(\alpha)$ generated by a block basic sequence of a basis other than (e_n), in particular a permutation of this basis. Thus we have our first extension of Rolewicz's result to more general power series spaces.

THEOREM 3. *If α is a nuclear exponent sequence of finite type which satisfies the condition*

$$\sup_n \frac{\alpha_{2n}}{\alpha_n} < \infty,$$

then $\Lambda_\infty(\alpha)$ is isomorphic to a subspace of $\Lambda_0(\alpha)$, in particular one which is generated by a block basic sequence of a permutation of the basis (e_n).

PROOF. First of all we show that we can assume that α is strictly increasing. Indeed, if (δ_n) is any sequence of positive numbers with $1 \leqq \delta_n \leqq 2$, chosen such that $\tilde{\alpha}_n = \alpha_n + \delta_n$ is strictly increasing, then $\tilde{\alpha}_n$ is again a nuclear exponent sequence of finite type and $\Lambda_0(\tilde{\alpha})$ is isomorphic to $\Lambda_0(\alpha)$ via the identity map. Moreover,

$$\begin{aligned}\sup_n \frac{\tilde{\alpha}_{2n}}{\tilde{\alpha}_n} &= \sup_n \frac{\alpha_{2n}+\delta_{2n}}{\alpha_n+\delta_n} \\ &\leqq \sup_n \frac{\alpha_{2n}}{\alpha_n} + 2\sup_n \frac{1}{\alpha_n} < \infty\end{aligned}$$

so all of our hypotheses still hold. Thus we may apply Lemma 2 to (α_n).

Let $\sigma\colon \mathbb{N}\times\mathbb{N}\to\mathbb{N}$ be the bijection given by

$$\sigma(j,n) = 2^{j-1}(2n-1)$$

and set $M = \sup_n \alpha_{2n}/\alpha_n$. Let n be fixed and for $k = 1,2,\cdots,n$ let j_n^k be the first positive integer such that

$$k^3 < \frac{\alpha_{\sigma(j_n^k,n)}}{\alpha_n}.$$

This is possible since α and $\sigma(\cdot,n)$ are increasing. There are two ways in which j_n^k could have been selected. First, suppose $j_n^k = 1$. Then we have

$$\frac{\alpha_{\sigma(j_n^k,n)}}{\alpha_n} = \frac{\alpha_{2n-1}}{\alpha_n} \leqq \frac{\alpha_{2n}}{\alpha_n} \leqq M \leqq Mk^3.$$

Otherwise, if $j_n^k > 1$ we have

$$\frac{\alpha_{\sigma(j_n^k-1,n)}}{\alpha_n} \leqq k^3$$

so that

$$\frac{\alpha_{\sigma(j_n^k,n)}}{\alpha_n} = \frac{\alpha_{\sigma(j_n^k,n)}}{\alpha_{\sigma(j_n^k-1,n)}} \cdot \frac{\alpha_{\sigma(j_n^k-1,n)}}{\alpha_n} \leqq Mk^3.$$

Thus in either case we have

$$k^3 < \frac{\alpha_{\sigma(j_n^k,n)}}{\alpha_n} < Mk^3, \qquad k = 1,2,\cdots,n.$$

Now we set $p_0 = 0$ and $p_n = \sigma(j_n^n, n)$. Obviously (j_n^k) and hence $\sigma(j_n^k,n)$ are non-decreasing with respect to k so we can apply Lemma 2 to assert the existence of scalars t_i^n, $i = 1,\cdots,p_n$ such that if

$$y_n = \sum_{i=1}^{p_n} t_i^n e_i,$$

then

$$q_n^k = \sigma(j_n^k,n), \qquad k = 1,\cdots,n.$$

Moreover, from the proof of Lemma 2, $t_i^n = 0$ if i is not one of $\sigma(i_n^k,n)$, $k = 1,\cdots,n$, and so we can write

$$y_n = \sum_{i=1}^{n} t^n_{\sigma(j_n^i,n)} e_{\sigma(j_n^i,n)}.$$

This shows that the sequence (y_n) is a block basic sequence of a permutation of the basis (e_n).

We complete the proof by applying Theorem 1 (iv). We have $\beta = \alpha$ and we take π to be the identity map so we obtain

$$k^3 < \gamma_n^k < Mk^3 \text{ for } k = 1,\cdots,n.$$

Hence, for each k,

$$\sup_n \gamma_n^k = \max\left\{\max_{1\leqq n<k} \gamma_n^k, Mk^3\right\} < \infty$$

and

$$\sup_n \varliminf_n \sum_{j=1}^{k} \frac{\gamma_n^j}{j^2} = \sup_k \varliminf_{n\geqq k} \sum_{j=1}^{k} \frac{\gamma_n^j}{j^2}$$

$$\geqq \sup_k \sum_{j=1}^{k} \frac{j^3}{j^2} = \infty.$$

Thus our theorem follows from Theorem 1.

REMARK 2. In view of Dragilev's theorem and the fact that multiplying a basis by a fixed sequence of nonzero scalars does not affect the nature of a block basic sequence, we can say that Theorems 2 and 3 consider all possible bases for the space $\Lambda_0(\alpha)$. Moreover, these two results imply that the fact that two bases are quasiequivalent does not imply that their block basic sequences generate isomorphic subspaces.

REMARK 3. It may seem that it is possible to improve Theorem 3 by replacing the permutation σ by some other permutation. However, an analysis of the proof shows that the crucial property of σ is the fact that for each j,

$$\sup_n \frac{\alpha_{\sigma(j,n)}}{\alpha_n} < \infty.$$

It was shown in [4] that the existence of any permutation with this property is equivalent to the fact that σ has this property and this is also equivalent to the hypothesis on α in Theorem 3. Thus it seems that this is the best result for the case $\alpha = \beta$ that can be obtained by this method.

However, if we drop the requirement that $\alpha = \beta$ then we can weaken the hypothesis on α and therefore get the following further extension of Rolewicz's theorem. In this case we once again get block basic sequences of (e_n).

THEOREM 4. *Let α be a nuclear exponent sequence of finite type. Assume that there exists a constant M and a sequence $I_n = [\phi_n, \lambda_n]$ of nonempty closed intervals of positive integers such that*

$$\frac{\alpha_{m+1}}{\alpha_m} \leqq M \text{ for } m \in I_n, \qquad n = 1,2,\cdots$$

and

$$\sup_n \frac{\alpha_{\lambda_n}}{\alpha_{\phi_n}} = \infty.$$

Then there exists a nuclear exponent sequence of finite type β such that $\Lambda_\infty(\beta)$ is isomorphic to a subspace of $\Lambda_0(\alpha)$ generated by a block basic sequence of (e_n).

PROOF. Our argument will be a variation of the proof of Theorem 3. As in that case we can use a simple perturbation argument to assume that α is strictly increasing. Moreover, by disregarding some of the intervals I_n and changing some others if necessary, we can easily arrange for the following properties to hold, where $f(n) = \alpha_{\lambda_n}/\alpha_{\phi_n}$:

$$1 < \phi_1, \ \lambda_n < \phi_{n+1}$$
$$1 \leqq f(n) \leqq f(n+1)$$

for $n = 1, 2, \cdots$ and

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^2} = \infty.$$

Now fix n and for $k = 1, 2, \cdots, n$ let q_n^k be the smallest positive integer such that

$$f(k)\alpha_{\phi_n} < \alpha_{q_n^k} \ .$$

Since $\phi_n > 1$, $f(k)\alpha_{\phi_n} \geqq \alpha_{\phi_n}$ and α is increasing, it follows that

$$q_n^k > \phi_n > 1 \ \text{for} \ k = 1, \cdots, n$$

and so we have

$$\alpha_{q_n^k - 1} \leqq f(k)\alpha_{\phi_n}.$$

Moreover, since $f(k) \leqq f(k+1)$ we have

$$q_n^k \leqq q_n^{k+1} \ \text{for} \ k = 1, 2, \cdots, n-1.$$

Next we have, for $k = 1, 2, \cdots, n$,

$$\alpha_{\lambda_n} = f(n)\alpha_{\phi_n} \geqq f(k)\alpha_{\phi_n}$$

so that

$$q_n^k \leqq \lambda_n \ \text{for} \ k = 1, 2, \cdots, n.$$

Finally we have for $k = 1, 2, \cdots, n$,

$$\alpha_{q_n^k} = \frac{\alpha_{q_n^k}}{\alpha_{q_n^k - 1}} \alpha_{q_n^k - 1} \leqq M f(k)\alpha_{\phi_n}.$$

Now let $p_0 = 0$ and $p_n = q_n^n$ for $n = 1, 2, \cdots$. Then we have, for $k = 1, 2, \cdots, n$,

$$p_{n-1} = q_{n-1}^{n-1} \leqq \lambda_{n-1} < \phi_n < q_n^k \leqq p_n,$$

so we can apply Lemma 2 with p_0, p_1 replaced by p_{n-1}, p_n to obtain $y_n \in \Lambda_0(\alpha)$ with

$$y_n = \sum_{i = p_{n-1}+1}^{p_n} t_i e_i$$

and

$$q_n^k = q_n^k(t_{p_{n-1}+1}, \cdots, t_{p_n}) \ \text{for} \ k = 1, 2, \cdots, n.$$

This is our block basic sequence. If we set $\beta_n = \alpha_{\phi_n}$ for $n = 1, 2, \cdots$ then since α is a nuclear exponent sequence of finite type, so is β and hence it is also of infinite type (Proposition 1(iv) and (vii)). To check that the subspace generated by (y_n) is isomorphic to $\Lambda_\infty(\beta)$ we take π to be the identity permutation and we have

$$\gamma_n^k = \frac{\alpha_{q_n^k}}{\beta_{\pi(n)}} = \frac{\alpha_{q_n^k}}{\alpha_{\phi_n}} \text{ for } k = 1,\cdots,n \text{ and } n = 1,2,\cdots$$

so it follows from the above inequalities that

$$f(k) < \gamma_n^k \leqq Mf(k) \text{ for } n \geqq k,\ n = 1,2,\cdots.$$

The desired result then follows from the properties of f and Theorem 1 (iv) so our theorem is proved.

REMARK 4. Obviously the conditions of Theorem 4 are satisfied whenever α satisfies the condition,

$$\sup_n \frac{\alpha_{n+1}}{\alpha_n} < \infty.$$

Since this condition is strictly weaker than the condition of Theorem 3, we have a proper generalization of that result.

REMARK 5. It is known that a complemented subspace of a power series space of finite type cannot be isomorphic to a power series space of infinite type [1, Corollary 2.9]. Thus Theorems 3 and 4 give examples of block basic sequences which generate subspaces that are not even isomorphic to complemented subspaces. Recently Lindenstrauss and Tzafriri [8] showed that the spaces l_p, $1 \leqq p < \infty$ and c_0 can be characterized as those infinite dimensional Banach spaces with unconditional bases with the property that every block basic sequence generates a complemented subspace.

Since all bases in nuclear Fréchet spaces are unconditional, it follows that if a basic sequence is a subsequence of a basis then it generates a complemented subspace. Hence, in view of the preceding comment it follows that Theorems 3 and 4 also provide examples of block basic sequences which cannot be extended to bases. It was recently shown [5] that the space ω (of all sequences of scalars) can be characterized as that nuclear Fréchet space with the property that every block basic sequence of every basis has a block extension to a basis.

We turn now to some negative results on the possibility of embedding a power series space of one type into one of another type. The first result in this direction was due to Zaharyuta [13] who showed that if α, β were nuclear exponent sequences of infinite and finite type respectively, then $\Lambda_0(\beta)$ is not isomorphic to a subspace of $\Lambda_\infty(\alpha)$. As we have seen in Theorems 3 and 4, this result does not remain true if the types are reversed, so we are led to ask for conditions on α (if any) which permit us to assert that $\Lambda_0(\alpha)$ contains no subspace isomorphic to a power series space of infinite type. If we look at Zaharyuta's argument we see that he obtains his result by proving that every linear continuous map from a finite type power series space to an infinite type power series space is compact. The conclusion about subspaces is then immediate. If we consider this statement in our context (that is, with "infinite" and "finite" interchanged), then we shall see that exactly the opposite result holds (Theorem 5). First we must prove a Lemma.

LEMMA 4. *Let α be a nuclear exponent sequence of finite type and $(\eta_\nu)_\nu$ a sequence of positive numbers. Then there exists a decomposition $\mathbb{N} = \bigcup_\nu \mathbb{N}_\nu$ of the positive integers into countably many pairwise disjoint subsequences such that if we define*

$$\beta_n = \frac{\alpha_n}{\eta_\nu},\ n \in \mathbb{N},\ \nu = 1,2,\cdots$$

then the sequence obtained by rearranging (β_n) into a nondecreasing sequence is a nuclear exponent sequence of infinite type.

PROOF. By Proposition 1 (ii) we can choose $\mathbb{N}_\nu$, $\nu = 2,3,\cdots$ such that $\mathbb{N}_1 = \mathbb{N} \sim \bigcup_{\nu=2}^{\infty} \mathbb{N}_\nu$ is infinite and

$$\sum_{n \in N_\nu} \frac{1}{\alpha_n} \leqq \frac{1}{2^\nu \eta_\nu},\ \nu = 2,3,\cdots.$$

Then

$$\sum_{n \notin N_1} \frac{1}{\beta_n} = \sum_{\nu=2}^{\infty} \sum_{n \in N_\nu} \frac{\eta_\nu}{\alpha_n} = \sum_{\nu=2}^{\infty} \eta_\nu \sum_{n \in N_\nu} \frac{1}{\alpha_n}$$
$$\leqq \sum_{\nu=2}^{\infty} \frac{1}{2^\nu} = \frac{1}{2} < \infty.$$

This implies that the set $(\beta_n)_{n \notin N_1}$ has at most finitely many elements less than any given number so it can be rearranged into a nondecreasing sequence which satisfies the condition of Proposition 1 (iii) so that it is a nuclear exponent sequence of infinite type. Since $\beta_n = \alpha_n / \eta_1$ for $n \in \mathbb{N}_1$ it follows from Proposition 1 (vi) that this is also a nuclear exponent sequence of infinite type. The Lemma then follows from Proposition 1(v).

THEOREM 5. *For every nuclear power series space X of finite type there* *exists a nuclear power series space Y of infinite type and a linear, continuous, noncompact map $T: Y \to X$.*

PROOF. We take $X = \Lambda_0(\alpha)$ and define $\eta_1 = 1$ and

$$\eta_\nu = \frac{\log \nu}{\log\left(\dfrac{\nu^2}{\nu^2 - 1}\right)} \quad \text{for } \nu = 2, 3, \cdots.$$

Then we obtain $\mathbb{N}_\nu$, $\nu = 1, 2, \cdots$ and β from Lemma 4 so that $Y = \Lambda_\infty(\beta)$ is a nuclear power series space of infinite type.

Let $\xi = (\xi_n)$ be the sequence given by

$$\xi_n = \nu^{\beta_n}\left(\frac{\nu + 1}{\nu}\right)^{\beta_{n \cdot \cdot}} \quad \text{for } n \in \mathbb{N}_\nu,\ \nu = 1, 2, \cdots,$$

and let T be the diagonal transformation determined by ξ. That is, $Ty = (\xi_n y_n)_n$ where y is any sequence. Using the closed graph theorem it follows that we need only show that $T(Y) \subset X$ and T is not compact. In order to do this it will be necessary to estimate the quantity

$$\xi_n^{1/\beta_n}\left(\frac{k}{k+1}\right)^{\alpha_n/\beta_n},\ n,\ k = 1, 2, \cdots.$$

We fix k and consider three cases.

If $\nu = k$ we have, for $n \in \mathbb{N}_\nu$,

$$\xi_n^{1/\beta_n}\left(\frac{k}{k+1}\right)^{\alpha_n/\beta_n} = \xi_n^{1/\beta}\left(\frac{\nu}{\nu+1}\right)^{\eta} = \nu = k.$$

If $\nu < k$ we have, for $n \in \mathbb{N}_\nu$,

$$\xi_n^{1/\beta_n}\left(\frac{k}{k+1}\right)^{\alpha_n/\beta_n} = \xi_n^{1/\beta_\cdot}\left(\frac{\nu}{\nu+1}\right)^{\eta_\nu}\left(\frac{k}{k+1}\cdot\frac{\nu+1}{\nu}\right)^{\eta\cdot}$$

$$\leqq \nu\left(\frac{\nu+1}{\nu}\right)^{\eta}.$$

If $\nu > k$ then, in particular $\nu > 1$ and we have, for $n \in N_\nu$,

$$\xi_n^{1/\beta_n}\left(\frac{k}{k+1}\right)^{\alpha_\cdot/\beta_\cdot} \leqq \xi_n^{1/\beta_n}\left(\frac{\nu-1}{\nu}\right)^{\eta_{\cdot\cdot}} = \nu\left(\frac{\nu^2-1}{\nu^2}\right)^{\eta}$$

$$= \nu\left(\frac{\nu^2-1}{\nu^2}\right)^{\log\nu/\log(\nu^2/\nu^2-1)} = 1.$$

Hence we have, for each k,

$$k \leqq \overline{\lim_n}\, \xi_n^{1/\beta_n} \left(\frac{k}{k+1}\right)^{\alpha_n/\beta_n} \leqq \max\left(k, 1, \max_{\nu<k} \nu\left(\frac{\nu+1}{\nu}\right)^{\eta_\nu}\right) < \infty,$$

so we may conclude,

$$\text{(4)} \qquad \text{for each } k, \ \sup_n \xi_n^{1/\beta_n} \left(\frac{k}{k+1}\right)^{\alpha_n/\beta_n} < \infty$$

$$\text{(5)} \qquad \sup_k \ \overline{\lim_n}\, \xi_n^{1/\beta_n} \left(\frac{k}{k+1}\right)^{\alpha_n/\beta_n} = \infty.$$

Now we are ready to prove that T has the desired properties.

$T(Y) \subset X$. Let $t = (t_n) \in Y$. Then for each k we have, from (4), an integer j such that

$$\xi_n^{1/\beta_n}\left(\frac{k}{k+1}\right)^{\alpha_n/\beta_n} \leqq j \text{ for all } n$$

and so we have, for all n,

$$\xi_n |t_n| \left(\frac{k}{k+1}\right)^{\alpha_n} = |t_n| j^{\beta_n} j^{-\beta_n} \xi_n \left(\frac{k}{k+1}\right)^{\alpha_n}$$
$$\leqq |t_n| j^{\beta_n}.$$

The fact that $t \in Y$ implies that $(t_n j^{\beta_n})_n \in l_1$ for each j and so

$$\left(\xi_n t_n \left(\frac{k}{k+1}\right)^{\alpha_n}\right)_n \in l_1$$

and since this holds for each k it follows that $T(Y) \subset X$.

T is not compact. Since Y is a nuclear Fréchet space, it it also a Montel space and hence it suffices to show that T maps every member of a fundamental system of neighborhoods of 0 for Y into a subset of X which is not bounded. We consider the fundamental system $(V_j)_j$ where

$$V_j = \left\{ t \in Y: \ \sum_n |t_n| j^{\beta_n} \leqq 1 \right\}, \quad j = 1, 2, \cdots.$$

Suppose that for some j, $T(V_j)$ is a bounded subset of X. This means that for each k there exists M_k such that

$$\sum_n |t_n|^{\beta_n} \leqq 1 \text{ implies } \sum_n \xi_n |t_n| \left(\frac{k}{k+1}\right)^{\alpha_n} \leqq M_k \text{ for all } k,$$

or

$$(t_n j^{\beta_n})_n \in l_1 \text{ implies } \left(\xi_n t_n \left(\frac{k}{k+1}\right)^{\alpha_n}\right)_n \in l_1 \text{ for all } k,$$

so that the sequence

$$\left(\xi_n j^{-\beta_n}\left(\frac{k}{k+1}\right)^{\alpha_n}\right)_n$$

defines a diagonal transformation of l_1 into itself and so it must be bounded, say by N_k, for each k. Thus we have, for each k,

$$\xi_n^{1/\beta_n}\left(\frac{k}{k+1}\right)^{\alpha_n/\beta_n} \leqq N_k^{1/\beta_n} j \text{ for all } n.$$

Taking the limit superior as n goes to infinity and using the fact (Proposition 1(ii)) that $\lim_n \alpha_n = \infty$ we obtain

$$\overline{\lim_n}\, \xi_n^{1/\beta_n}\left(\frac{k}{k+1}\right)^{\alpha_n/\beta_n} \leqq j,$$

and this is true for each k independently of j. Obviously this contradicts (5) so the map T is not compact and the theorem is proved.

Thus the method of Zaharyuta is totally unavailable for showing the non-existence of subspaces of $\Lambda_0(\alpha)$ which are isomorphic to some $\Lambda_\infty(\beta)$. Indeed, results like Theorem 3 and 4 along with Theorem 5 might suggest that the opposite is the case and that every $\Lambda_0(\alpha)$ contains such a subspace. For our last result we show that this is not the case.

THEOREM 6. *Let α be a nuclear exponent sequence of finite type which satisfies the condition*

$$\lim_n \frac{\alpha_{n+1}}{\alpha_n} = \infty.$$

Then $\Lambda_0(\alpha)$ has no subspace isomorphic to a power series space of infinite type.

PROOF. Suppose that X is a subspace of $\Lambda_0(\alpha)$ which is isomorphic to a nuclear power series space of infinite type. Then X has a basis (x_n) which is then a basic sequence in $\Lambda_0(\alpha)$. If we apply a variation of a theorem of C. Bessaga and A. Pełczyński [2, Theorem 1], then we may conclude that there is a basic sequence (y_n) in $\Lambda_0(\alpha)$ which generates a subspace Y isomorphic to X and such that each y_n is a finite linear combination of the elements of the coordinate basis (e_n). The variation consists of two changes. First, the result of Bessaga and Pełczyński is for bases, but no change in the argument is required to obtain it for basic sequences.

Secondly, they do not explicitly show that X is isomorphic to Y but this is implicit in the proof.

Thus we have the situation described in the beginning of this paper and set up for an application of Theorem 1. We can write $Y = \Lambda_\infty(\beta)$ and we shall obtain a contradiction for two separate cases. Let k_0 be the integer given in Proposition 2.

First consider the case in which there exists $k \geqq k_0$ and an infinite subset $\mathbb{N}_0 \subset \mathbb{N}$ such that $q_n^{k+1} > q_n^k$ for all $n \in \mathbb{N}_0$. Then using the hypothesis on α and the second conclusion in Proposition 2 we have

$$\lim_{n \in N_0} \frac{\gamma_n^{k+1}}{\gamma_n^k} = \lim_{n \in N_0} \frac{\alpha_{q_n^{k+1}}}{\alpha_{q_n^k}} \geqq \lim_{n \in N_0} \frac{\alpha_{q_n^{k+1}}}{\alpha_{q_n^k}} = \infty.$$

But this fact along with the first conclusion in Proposition 2 implies that $\lim_{n \in N_0} \gamma_n^{k+1} = \infty$ and this contradicts Theorem 1 (iv) (a).

In view of Lemma 1, the only other possibility is that for each $k \geqq k_0$ there exists an integer n_k such that $q_n^k = q_n^{k+1}$ for all $n \geqq n_k$ and this implies that $\gamma_n^k = \gamma_n^{k+1}$ for all $n \geqq n_k$. Therefore $\overline{\lim}_n \gamma_n^k = \overline{\lim}_n \gamma_n^{k_0}$ for all $k \geqq k_0$ so we have, using Theorem 1 (iv) (a),

$$\begin{aligned}\sup_k \overline{\lim_n} \sum_{j=1}^k \frac{\gamma_n^j}{j^2} &= \max\left(\sup_{k<k_0} \overline{\lim_n} \sum_{j=1}^k \frac{\gamma_n^j}{j^2}, \sup_{k \geqq k_0} \overline{\lim_n} \sum_{j=1}^k \frac{\gamma_n^j}{j^2}\right) \\ &\leqq \max\left(\sup_{k<k_0} \sum_{j=1}^k \frac{\overline{\lim}_n \gamma_n^j}{j^2}, \sup_{k \geqq k_0} \sum_{j=1}^k \frac{\overline{\lim}_n \gamma_n^j}{j^2}\right) \\ &= \sup_{k \leqq k_0} \sum_{j=1}^k \frac{\overline{\lim}_n \gamma_n^j}{j^2} < \infty\end{aligned}$$

which contradicts Theorem 1 (iv) (b), so our theorem is proved.

REMARK 6. It is easy to see that the hypotheses of Theorems 4 and 6 do not include all possibilities. For example, the sequence

$$\alpha_1, \alpha_1, \alpha_2, \alpha_2, \alpha_3, \alpha_3, \cdots$$

where $\alpha_n = 2^{2^n}$ is a nuclear exponent sequence of finite type which fails to satisfy either condition. Thus we are led to the following question.

PROBLEM. Characterize those nuclear exponent sequences of finite type α for which $\Lambda_0(\alpha)$ contains a subspace isomorphic to a power series space of infinite type.

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